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Projective Quantales: A General View

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Abstract Following B. Banaschewski's work on so-called **K**-flat projective frames, we treat their non-commutative generalization, i.e. **K**-flat projective quantales. Besides a categorical definition of a **K**-flat projective quantale we also give an elementary characterization in terms of a binary relation on the quantale, and we prove that **K**-flat projective quantales are precisely the coalgebras for a certain comonad on the category of quantales. Finally we define and study a natural notion of **K**-coherentness for quantales.

Keywords Quantale · K-flat projectivity · K-command

1 Introduction

In this paper we present a general view wrt. the projectivity notion in the category of quantales.

The issues for this paper were mainly the paper [1] where Banaschewski establishes both the internal and the external characterization of "projective" objects in the category of frames and the article [3] where the characterization of quantales projective wrt. regular epimorphisms such that their right adjoint is a semigroup homomorphism is given. There is also a study of projective objects in the theory of quantized functional analysis [2] generalizing the Grothendieck's characterization of $l^1(I)$ spaces for a discrete set I. We will generalize both the results in [1] and [3].

The paper is organized as follows. Section 1 follows closely the approach introduced in [1]. It starts with a brief introduction to the notion of **K**-flat projectivity. For each quantale L we then introduce a binary relation $\triangleleft_L$ on it and we establish basic properties of this relation. We will prove that the **K**-flat projective quantales are exactly such quantales with

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each element a approximated by the elements x, $x \lhd_L a$ and the relation $\lhd_L$ being stable wrt. the quantale multiplication on L. Further on, we introduce the notion of a **K**-comonad and characterize **K**-flat projective quantales as such quantales having a coalgebra structure for the **K**-comonad. In Sect. 2, we define and study a notion of **K**-coherentness for quantales, and give a general version of coreflection results from [3].

For a general overview on category we recommend [4], for facts concerning quantales in general we refer to [6].

2 K-Flat Projective Quantales

Generalizing B. Banaschewski's work in [1] on frames to the non-commutative case, our setting here is the category **O***Sgr* of partially ordered semigroups in which we consider subcategories **K** containing the category **Q***uant* of quantales reflectively, subject to a very simple condition:

(C) *For any $\varphi : A \to L$ in* **K** *where L is a quantale and A arbitrary, the corestriction of φ to any subquantale of L containing the image of φ also belongs to* **K**.

We refer to this by saying that **K** is *corestrictive* over **Q***uant*. We now define a quantale L to be **K**-*flat projective* if L is projective in **Q***uant* relative to the onto quantale homomorphisms $h : L \to M$ for which the right adjoint $h_* : M \to L$ ($h(a) \leq b$ iff $a \leq h_*(b)$) belongs to **K**.

Since **K** contains the category **Q***uant* reflectively we have, for any object A from **K**, the universal map in **K** to quantales $\eta_A : A \to FA$ and, correspondingly, for any quantale L, the quantale homomorphism $\varepsilon_L : FL \to L$ such that $\varepsilon_L \circ \eta_L = \mathrm{id}_L$. Further, $\leq$ stands for the usual argumentwise partial order of maps between partially ordered sets, which is evidently preserved by the composition of maps in **K**.

Lemma 1 (1) *Each FA is generated by the image of η_A.*

(2) $\mathrm{id}_{FL} \leq \eta_L \circ \varepsilon_L$ *for any quantale L.*

(3) η_A *reflects order that is $f \circ \eta_A \leq g \circ \eta_A$ implies $f \leq g$ for any quantale homomorphisms $f, g : FA \to L$.*

(4) *For any quantale L, if $h : L \to FL$ is a right inverse to $\varepsilon_L : FL \to L$ then $h \circ \varepsilon_L \leq \mathrm{id}_{FL}$.*

Proof The proof is a direct reformulation of the proof of Lemma 1 in [1]. □

Lemma 2 *Let $g : A \to B$ be a morphism in* **K** *and $f : K \to L$ a morphism in* **Q***uant. Then the following diagrams*

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & FA \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Fg} \\ B & \xrightarrow{\eta_B} & FB \end{array} \qquad \text{and} \qquad \begin{array}{ccc} FK & \xrightarrow{\varepsilon_K} & K \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FL & \xrightarrow{\varepsilon_L} & L \end{array}$$

commute in the respective categories. Moreover, F preserves the partial order of maps and $F\eta_A$ is left adjoint to ε_{FA} in **Q***uant.*

Proof The first statement is straightforward and the proof of the second statement goes the same way as the proof of Lemma 2 in [1]. □

Remark 3 As in [1], we have that $\varepsilon_L \circ \eta_L = \mathrm{id}_L$ and (2) from Lemma 1 imply that η_L is right adjoint to ε_L. Similarly, if $\varepsilon_L \circ h = \mathrm{id}_L$ for some $h : L \to FL$ then h is left adjoint to ε_L by (4) from Lemma 1 and consequently unique.

The reflectiveness of **Q***uant* in **K** determines a binary relation on each quantale L as follows:

$$x \lhd_L a \quad \text{iff} \quad a \le \varepsilon_L(b) \text{ implies } \eta_L(x) \le b, \ \text{ for all } b \in FL.$$

Lemma 4 *Let K, L be quantales. Then, for all $x, y, u, v \in K$ and for any quantale homomorphism $f : K \to L$, we have*

(1) *$x \le y \lhd_K u \le v$ implies $x \lhd_K v$ and $x \le v$,*
(2) *if f is an isomorphism of quantales then $x \lhd_K u$ if and only if $f(x) \lhd_L f(u)$.*

Proof The proof is straightforward. □

Now we may go on our first result concerning **K**-flat projectivity.

Theorem 5 *The following are equivalent for any quantale L.*

(1) *L is **K**-flat projective.*
(2) *ε_L has a right inverse.*
(3) *L is a retract of some FA, $A \in$ **K**.*
(4) *For each $a \in L$, $a = \bigvee\{x \in L | x \lhd_L a\}$; further $x \cdot y \lhd_L a \cdot b$ whenever $x \lhd_L a$ and $y \lhd_L b$.*

Proof (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (1) and (2) $\Longrightarrow$ (4) can be proved as in [1], Proposition 1.
(4) $\Longrightarrow$ (2). Let $h_L : L \to FL$ be the set map defined by $h_L(a) = \bigvee\{\eta_L(x) : x \lhd_L a\}$. Then as in [1], Proposition 1 we have that $\varepsilon_L \circ h_L = \mathrm{id}_L$, $(h_L \circ \varepsilon_L)(b) = \bigvee\{\eta_L(x) : x \lhd_L \varepsilon_L(b)\} \le b$, h_L preserves arbitrary joins and, for any $a, b \in L$, $h_L(a) \cdot h_L(b) \le h_L(a \cdot b)$. To check the converse inequality, we have

$$\begin{aligned} h_L(a \cdot b) &= h(\varepsilon_L(h_L(a)) \cdot \varepsilon_L(h_L(b))) = h_L(\varepsilon_L(h_L(a) \cdot h_L(b))) \\ &= (h_L \circ \varepsilon_L)(h_L(a) \cdot h_L(b)) \le \mathrm{id}_{FL}(h_L(a) \cdot h_L(b)) = h_L(a) \cdot h_L(b). \end{aligned}$$

In all, this shows that h_L is a quantale homomorphism, right inverse to ε_L. □

As in [1], there is a further characterization of **K**-flat projectivity involving the comonad in **Q***uant* determined by the reflection functor F.

The comonad determined by F (viewed as an endofunctor of **Q***uant*) is $(F, \varepsilon, F\eta)$, and its coalgebras are the pairs (L, h_L) where the structure map $h_L : L \to FL$ satisfies the conditions (see [4]):

$$\text{(U)} \quad \varepsilon_L \circ h_L = \mathrm{id}_L \quad \text{and} \quad \text{(A)} \quad (Fh_L) \circ h_L = (F\eta_L) \circ h_L.$$

Given that it is entirely determined by the extension **K** of **Q***uant*, we call this comonad the **K**-comonad. The desired result then is

Proposition 6 *A quantale L is* **K***-flat projective iff it has a coalgebra structure for the* **K***-comonad.*

Proof The proof follows the proof of Proposition 2 in [1]. □

Similarly as in [1], let us consider the following way of specifying subcategories of $\mathbf{O}Sgr$. For each $A \in \mathbf{O}Sgr$, let $\mathcal{S}A$ be a collection of subsets of A such that $\{a \cdot t : t \in S\}$ and $\{t \cdot a : t \in S\}$ belongs to $\mathcal{S}A$ for each $a \in A$ and $S \in \mathcal{S}A$, and for each $f : A \to B$ in $\mathbf{O}Sgr$, $f[S] \in \mathcal{S}B$ whenever $S \in \mathcal{S}A$.

Further, let $\mathbf{S}$ be the subcategory of $\mathbf{O}Sgr$ consisting of all A such that $\bigvee S$ exists for each $S \in \mathcal{S}A$, $a \cdot \bigvee S = \bigvee\{a \cdot t : t \in S\}$ and $\bigvee S \cdot a = \bigvee\{t \cdot a : t \in S\}$, the maps being the $\mathbf{O}Sgr$-homomorphisms which preserve all $\bigvee S$, $S \in \mathcal{S}A$.

Let us recall the following familiar, particularly significant examples: for each $A \in \mathbf{O}Sgr$, $\mathcal{S}A$ consists of

(1) no S, and $\mathbf{S} = \mathbf{O}Sgr$,
(2) $\emptyset$, and $\mathbf{S}$ is the category of lower-bounded partially ordered semigroups.
(3) all finite subsets, and $\mathbf{S}$ is the category of m-semilattices.
(4) All (at most) countable subsets, and $\mathbf{S}$, the category of σm-semilattices.
(5) all updirected subsets, and $\mathbf{S} = \mathbf{P}r\mathbf{Q}uant$, the category of prequantales, and
(6) all subsets, and $\mathbf{S} = \mathbf{Q}uant$.

Recall that any category $\mathbf{S}$ of this kind trivially contains $\mathbf{Q}uant$ and is evidently corestrictive over quantales. But as in [1] we have more:

Proposition 7 *For any* $\mathbf{S}$, $\mathbf{Q}$*uant is reflective in* $\mathbf{S}$.

Proof As in [1], we will give an explicit description of the quantale reflection in $\mathbf{S}$. For any $A \in \mathbf{S}$, let $\mathcal{D}(A)$ be the quantale of all downsets of A, that is, the $U \subseteq A$ such that $a \in U$ whenever $a \le b$ and $b \in U$ (which includes $U = \emptyset$), and γA the closure system in $\mathcal{D}(A)$ consisting of all downsets U for which $S \subseteq U$ and $S \in \mathcal{S}A$ implies $\bigvee S \in U$.

Note that the principal downsets $\downarrow a = \{x \in A : x \le a\}$ belong to γA, giving rise to a map $\sigma_A : A \to \gamma A$ taking a to $\downarrow a$. For any $A \in \mathbf{S}$,

(i) γA is a quantale, and
(ii) $\sigma_A : A \to \gamma A$ is the universal map in $\mathbf{S}$ to quantales.

The proof of (i) and (ii) follows the proof of Proposition 3 in [1]. □

As in [1], in the examples listed above, the reflection γA consists of the following downsets U of A:

(1) all downsets U,
(2) all downsets U containing 0,
(3) the ideals of A,
(4) the σ-ideals of A,
(5) the Scott-closed downsets U, and
(6) the principal downsets $\downarrow a$.

For general $\mathbf{S}$, by the same arguments as in [1], we have that the adjunction map $\varepsilon_L : \gamma L \to L$, for any quantale L, is the join map.

On the other hand, as in [1], for any $A \in \mathbf{S}$, $x \in A$, and $U \in \gamma A$, $\eta_A(x) = \downarrow x \leq U$ iff $x \in U$, and hence the relation $\lhd_L$ now has the following concrete form: $x \lhd_L a$ iff $a \leq \bigvee U$ implies $x \in U$, for all $U \in \gamma L$.

3 K-Coherent Quantales

An element $a \in L$ is called **K**-*compact* if $a \lhd_L a$. Let $c_{\mathbf{K}}(L)$ denote all the **K**-compact elements of L. If L satisfies the following conditions:

(1) for all $a \in L$, $a = \bigvee\{c \in c_{\mathbf{K}}(L) : c \leq a\}$
(2) if $c_1, c_2 \in c_{\mathbf{K}}(L)$, then $c_1 \cdot c_2 \in c_{\mathbf{K}}(L)$,

then L is called **K**-*coherent quantale*.

Lemma 8 *Let* **K** *has equalizers coinciding with the equalizers in sets and let L be* **K**-*flat projective. Then $c_{\mathbf{K}}(L)$ is in* **K**.

Proof Evidently, $c_{\mathbf{K}}(L) = equal(\eta_L, h_L)$ by the Theorem 5. □

Note that **K** has equalizers coinciding with the equalizers in sets e.g. in the case when **K** is an equationally presentable category i.e. its objects can be prescribed by (a proper class of) operations and equations (see [5]).

Proposition 9 *For any $A \in \mathbf{K}$, FA is* **K**-*coherent. Moreover, $c_{\mathbf{K}}(FA)$ contains the set $\eta_A(A)$.*

Proof Let $a \in A$. By the Lemma 2, $F\eta_A$ is left adjoint to ε_{FA} in **Q***uant* and $F\eta_A \circ \eta_A(a) = \eta_{FA} \circ \eta_A(a)$. Hence $h_{FA} = F\eta_A$ and $h_{FA}(\eta_A(a)) = \eta_{FA}(\eta_A(a))$. Therefore $c_{\mathbf{K}}(FA) \supseteq \eta_A(A)$. Since FA is **K**-flat projective and $\eta_A(A)$ is a join-base of FA, FA is **K**-coherent. □

Proposition 10 *Let* **W** *be any subcategory of* **K**-*flat projective quantales containing the class $F\mathbf{K}$, with those quantale homomorphisms that preserve the relation $\lhd$. Then* **W** *is coreflective in* **Q***uant, with the coreflection functor F and the coreflection map $\varepsilon_L : FL \to L$.*

Proof The proof follows the proof of Proposition 3 in [3]. □

We now turn to the list of examples at the end of the previous section; we keep the notations introduced there. Since **S** is equationally presentable we have the following:

Proposition 11 *For any $A \in \mathbf{S}$, γA is* **S**-*coherent. Moreover, $A \simeq \sigma_A(A) = c_{\mathbf{S}}(\gamma A)$.*

Proof Recall that the reflection γA consists of all downsets U of A for which $S \subseteq U$ and $S \in \mathcal{S}A$ implies $\bigvee S \in U$.

The first part follows from the Proposition 9. We know also that $\sigma_A(A) \subseteq c_{\mathbf{S}}(\gamma A)$. Let $U \in \gamma A$. Then $\bigcup_{u \in U} \sigma_{\gamma A}(\downarrow u) \in \gamma(\gamma A)$.

Namely, let $\{I_\alpha : \alpha \in \Lambda\} \in \mathcal{S}(\gamma A)$, $\{I_\alpha : \alpha \in \Lambda\} \subseteq \bigcup_{u \in U} \sigma_{\gamma A}(\downarrow u)$. Then, for any $\alpha \in \Lambda$, $I_\alpha \in \sigma_{\gamma A}(\downarrow u_\alpha)$ for some $u_\alpha \in U$. Hence $I_\alpha \subseteq \downarrow u_\alpha$. Let us put $v_\alpha = \bigvee I_\alpha \leq u_\alpha$. Since $\varepsilon_{\gamma A} : \gamma(\gamma A) \to \gamma A$ is in **O***Sgr*, we have that $\varepsilon_{\gamma A}[\{I_\alpha : \alpha \in \Lambda\}] = \{v_\alpha : \alpha \in \Lambda\} \in \mathcal{S}A$. Let $v = \bigvee\{v_\alpha : \alpha \in \Lambda\}$. Then $v \in U$ and therefore $I_\alpha \subseteq \downarrow v$ and also $I_\alpha \in \sigma_{\gamma A}(\downarrow v)$ for any

$\alpha \in \Lambda$. This gives us that $\{I_\alpha : \alpha \in \Lambda\} \subseteq \sigma_{\gamma A}(\downarrow v)$. Hence $\bigvee\{I_\alpha : \alpha \in \Lambda\} \in \sigma_{\gamma A}(\downarrow v) \subseteq \bigcup_{u\in U} \sigma_{\gamma A}(\downarrow u)$.

Now, let $U \lhd_{\gamma A} U = \varepsilon_{\gamma A}(\bigcup_{u\in U} \sigma_{\gamma A}(\downarrow u))$. Then $\sigma_{\gamma A}(U) \subseteq \bigcup_{u\in U} \sigma_{\gamma A}(\downarrow u)$. Hence $U \in \bigcup_{u\in U} \sigma_{\gamma A}(\downarrow u)$ and therefore $U \in \sigma_{\gamma A}(\downarrow u)$ for some $u \in U$. This implies that $U \subseteq \downarrow u \subseteq U$ i.e. $U = \downarrow u$. It follows that $c_{\mathbf{S}}(\gamma A) \subseteq \sigma_A(A)$. □

4 Some Closing Comments

We can obtain results on **K**-flat projectivity and **K**-coherence for unital, or left-sided, or right-sided or idempotent quantales, or any combination of these properties: simply by substituting the enveloping category **O***Sgr* of partially ordered semigroups by the category of unital, or left-sided, or right-sided or idempotent partially ordered semigroups (or the appropriate combination). A frame is precisely a two-sided, idempotent quantale, hence the theory of **K**-flat projective frames in [1] is a special case of the more general results outlined here.

In future work, we hope that we will be able to improve the obtained results in a more general form for sup-algebras satisfying prescribed identities. Another related question is to obtain a full characterization of projective quantales, a problem that was started in [3].

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